AIAA 82-4061

Generation of Orthogonal and Nearly Orthogonal Coordinates with Grid Control near Boundaries

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Abstract

A NUMERICAL procedure is presented for the generation of boundary-fitted curvilinear coordinates with controllable mesh spacing and orthogonality. The technique is based on a straightforward two-step procedure involving the solution of Poisson's equation. The method is capable of generating orthogonal grids with partial control of the mesh spacing, or nearly orthogonal grids with strict control of the mesh spacing. The inhomogeneous terms in the Poisson equations are automatically adjusted during the grid generation. The technique is applicable to two-dimensional simply-connected regions typical of airfoils, cascades, diffusers, and inlets.

Contents

The numerical solution of flow problems in irregular regions is facilitated by the introduction of boundary-fitted curvilinear coordinates (see, for example, Ref. 1). Although the desirable features of a curvilinear mesh depend, to a large extent, upon the particular application, the following guidelines can be suggested. First, it is oftentimes desirable for the coordinates to be orthogonal or nearly orthogonal, particularly within boundary layer regions at high Reynolds number. Second, the method should provide direct control of the mesh spacing in order to concentrate grid points in areas where gradients of the flow variables are large. Third, the procedure should be efficient and *automatic* in the sense that iterative adjustment of parameters by the user is not required.

Considerable progress has been achieved in recent years in the generation of boundary-fitted coordinates. Various techniques are presented in Ref. 2. None of the methods based on elliptic partial differential equations provides all of the desirable features discussed above.[3] In this work a new technique[3,4] based on the solution of Poisson's equation was developed which exhibits all of the desired features. The procedure is presented for two-dimensional singly-connected regions, as shown in Figs. 1 and 2, where Γ_1 and Γ_2 are two arbitrarily oriented straight lines, and Γ_3 and Γ_4 are arbitrary curvilinear boundaries.

The technique consists of an intermediate and a final transformation. In the intermediate transformation, an orthogonal mesh is generated with a given distribution of the $\xi=$const lines on Γ_3. The final transformation can be used in two ways. First, an orthogonal grid can be obtained with the desired distribution of the mesh points along Γ_1 and Γ_3. Second, a nearly orthogonal grid can be generated with the desired distribution of mesh points on Γ_1, Γ_2, and Γ_3 and with direct control of the normal distance of the $\eta=$const lines from the boundaries Γ_3 and Γ_4. The method is fully *automatic* and does not require a priori knowledge of forcing functions (i.e., source terms in the Poisson equations).

Received March 18, 1981; synoptic received July 29, 1981. Full paper available from National Technical Information Service, Springfield, Va. 22151 at the standard price (available upon request).

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Intermediate Transformation

First, an orthogonal grid is generated with a user-specified distribution of the ξ lines on Γ_3. The transformation $[\xi(x,y), \chi(x,y)]$ satisfies the Poisson equations[3,4]

$$\xi_{xx}+\xi_{yy}=\phi(\xi,\chi)=\frac{1}{h_\xi h_\chi}\frac{\partial}{\partial\xi}\left(\frac{h_\chi}{h_\xi}\right) \tag{1a}$$

$$\chi_{xx}+\chi_{yy}=0 \tag{1b}$$

where $h_\xi=(x_\xi^2+y_\xi^2)^{1/2}$ and $h_\chi=(x_\chi^2+y_\chi^2)^{1/2}$ are the scale factors. The boundary conditions for ξ are $\xi=0$ on Γ_1, $\xi=1$ on Γ_2, $\xi=f_3(t)$ on Γ_3 (where t is the arc length along Γ_3 measured from Γ_1), and $\partial\xi/\partial n=0$ on Γ_4. The boundary condition on Γ_3 simply indicates that the user can distribute the ξ lines along Γ_3 in any arbitrary monotonic fashion, and no analytic expression for $f_3(t)$ is required. In the boundary condition along Γ_4, n denotes the normal to the boundary. The boundary conditions for χ are $\partial\chi/\partial n=0$ on Γ_1 and Γ_2, $\chi=0$ on Γ_3, and $\chi=1$ on Γ_4. The forcing function ϕ in Eq. (1a) results from the general expression for the Laplacian in orthogonal curvilinear coordinates. The intermediate mesh is obtained by inverting[3] Eqs. (1) and the boundary conditions, and solving for $[x(\xi,\chi), y(\xi,\chi)]$ in a uniform rectangular grid in the (ξ,χ) plane using point successive over-relaxation (SOR). During the solution, the forcing function ϕ is updated and the mesh points are redistributed along Γ_1, Γ_2, and Γ_4 to satisfy the boundary conditions.[3]

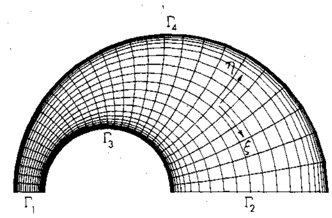

Fig. 1 Nearly orthogonal grid for the region between two non-concentric circles with a user-specified distribution of mesh points along Γ_1, Γ_2, and Γ_3.

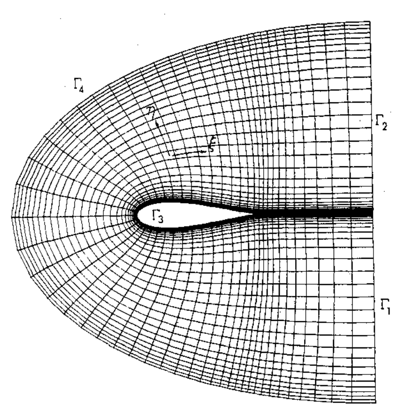

Fig. 2 C-grid about a symmetric Joukowski airfoil with a user-specified distribution of mesh points along Γ_1, Γ_2, and Γ_3.

The purpose of the intermediate transformation is to *efficiently* obtain the forcing function ϕ and the distribution of ξ lines along Γ_4 which are consistent with orthogonality. This distribution of ξ lines along Γ_3 and Γ_4 are then prescribed as Dirichlet boundary conditions in the solution of the final transformation. Also, the intermediate grid is employed in the evaluation of the forcing functions for the final mesh (see next section) and in the initial guess for the numerical solution of the final transformation equations (the intermediate transformation must be stored for this purpose). Our experience indicates that in the intermediate transformation a relatively coarse grid in the χ direction may be employed.

Final Transformation

The intermediate transformation incorporates the desired characteristics of orthogonality and arbitrary specification of the ξ lines along Γ_3. However, it does not provide control of the mesh spacing in the χ direction. The final transformation is then introduced in order to obtain either an orthogonal or a nearly orthogonal grid with controllable mesh spacing in both coordinate directions. The final grid is generated as follows. Consider the transformation $\chi=\chi(\eta)$. Then Eqs. (1) become[4]

$$\xi_{xx}+\xi_{yy}=P(\xi,\eta)=\frac{1}{h_\xi h_\eta}\frac{\partial}{\partial\xi}\left(\frac{h_\eta}{h_\xi}\right) \tag{2a}$$

$$\eta_{xx}+\eta_{yy}=Q(\xi,\eta)=\sigma(\eta)\,(\eta_x^2+\eta_y^2) \tag{2b}$$

where $\sigma(\eta)=-(\mathrm{d}^2\chi/\mathrm{d}\eta^2)/(\mathrm{d}\chi/\mathrm{d}\eta)$, and h_ξ and h_η are the scale factors. The boundary conditions for ξ on Γ_1, Γ_2, and Γ_3 are the same as in the intermediate transformation. The boundary condition on Γ_4 becomes $\xi=f_4(t)$, where $f_4(t)$ denotes the distribution of the ξ lines along Γ_4 obtained in the intermediate transformation and t is the arc length along Γ_4. The boundary conditions on η for orthogonal grids are $\eta=f_1(s)$ on Γ_1 (where s is the arc length along Γ_1 measured from Γ_3), $\partial\eta/\partial n=0$ on Γ_2, $\eta=0$ on Γ_3, and $\eta=1$ on Γ_4. The boundary condition on Γ_1 simply indicates that the η lines are distributed along Γ_1 as desired by the user. For nearly orthogonal grids with direct control of the normal distance of the $\eta=$const lines, the boundary condition on Γ_2 becomes $\eta=f_2(s)$ indicating a user-specified distribution of η lines. The inverted equations are

$$\alpha x_{\xi\xi}-2\beta x_{\xi\eta}+\gamma x_{\eta\eta}=-J^2(x_\xi P+x_\eta Q) \tag{3a}$$

$$\alpha y_{\xi\xi}-2\beta y_{\xi\eta}+\gamma y_{\eta\eta}=-J^2(y_\xi P+y_\eta Q) \tag{3b}$$

where $\alpha=x_\eta^2+y_\eta^2$, $\beta=x_\xi x_\eta+y_\xi y_\eta$, $\gamma=x_\xi^2+y_\xi^2$, and $J=x_\xi y_\eta-x_\eta y_\xi$. Equations (3) are solved using point SOR.

The forcing function $P(\xi,\eta)$ is determined from the intermediate transformation according to $P(\xi,\eta)=\phi(\xi,\chi(\eta))$. In general, the function $\chi(\eta)$ is not known a priori. However, since ϕ usually varies smoothly with χ, satisfactory results are obtained when the initial guess for x and y [used in the numerical solution of Eqs. (3)] and linear interpolation along the ξ lines (obtained in the intermediate transformation) are employed. For orthogonal grids, the function $P(\xi,\eta)$ may be updated during the solution, although experience with highly curved and clustered grids indicates that such adjustment is unnecessary. For *nearly* orthogonal grids with controlled spacing of the η lines, iterative adjustment of P in the final transformation according to the *orthogonal* expression of Eq. (2a) is inconsistent and is therefore not performed.

The forcing function Q is determined as follows. Application of the condition of local orthogonality $\beta=0$ in Eqs. (3) gives $Q=\gamma S/J^2$ where

$$S=T-(y_\eta/x_\xi)(R/\gamma)\ \text{if}\ x_\xi\neq 0,\quad T+(x_\eta/y_\xi)(R/\gamma)\ \text{if}\ y_\xi\neq 0 \tag{4}$$

$$T=-(x_\eta x_{\eta\eta}+y_\eta y_{\eta\eta})/(x_\eta^2+y_\eta^2) \tag{5}$$

$$R=x_\xi y_{\xi\xi}-y_\xi x_{\xi\xi} \tag{6}$$

It can be simply shown that $S=\sigma$ [see Eq. (2b)]. For an orthogonal grid, therefore, S is a function of η alone, and is evaluated at $\xi=0$ according to Eqs. (4-6) with the ξ derivatives approximated using the intermediate solution and linear interpolation. These derivatives can be updated during the solution, although experience indicates that such adjustment is unnecessary. For a nearly orthogonal grid with controlled spacing of the η lines, the function R is obtained from the intermediate mesh using linear interpolation in ξ. If the distribution of the η lines is specified in terms of the distance $s(\xi,\eta)$ measured from Γ_3 along the ξ lines, then

$$s_\eta^2=x_\eta^2+y_\eta^2,\quad s_\eta s_{\eta\eta}=x_\eta x_{\eta\eta}+y_\eta y_{\eta\eta} \tag{7}$$

From Eqs. (5) and (7), the function T becomes $T=-s_{\eta\eta}/s_\eta$. Therefore, T can be determined for any arbitrary distribution of the η lines. In this research, the η lines were distributed exponentially near Γ_3 and Γ_4 and uniformly in between. The resulting expression for T is $T=-C_1/\eta_1$ for $0\leq\eta\leq\eta_1$, $T=0$ for $\eta_1<\eta<\eta_2$, and $T=C_2/(1-\eta_2)$ for $\eta_2\leq\eta\leq 1$, where C_1, C_2 and η_1, η_2 are slowly varying functions of ξ and are determined[3] to provide control of the normal distance of selected η lines from Γ_3 and Γ_4.

Results

Several examples of orthogonal and nearly orthogonal grids have been presented in detail in Refs. 3 and 4. In Fig. 1 a nearly orthogonal mesh for the curved region between two nonconcentric circles is shown. The above expression for T was used with $\eta_1=7/15$ and $\eta_2=2/3$. The values of C_1 and C_2 at each ξ line were determined by the requirements that the normal distance between Γ_3 and the next η line be constant and the normal distance between Γ_4 and the next η line increase smoothly to four times its value on Γ_1 over the length of the flow region. The average deviation from orthogonality was only 0.7 deg. In Fig. 2, a C-grid about a symmetric Joukowski airfoil with 20% thickness ratio is shown. The boundary Γ_4 was chosen to be elliptical. The ξ lines were arbitrarily distributed along Γ_3 with clustering near the leading and trailing edges and a geometric stretching along the cut. The function T employed $\eta_1=7/15$ and $\eta_2=2/3$, with C_1 and C_2 determined to provide a constant normal distance from Γ_3 and Γ_4 to their adjacent η lines. The average deviation from orthogonality was only 2.5 deg. An orthogonal grid was also generated[3] for the same region with a resulting average deviation from orthogonality of only 0.6 deg.

The previous examples illustrate the capabilities of the present technique for the *automatic* generation of orthogonal and nearly orthogonal curvilinear coordinates with direct control of the mesh spacing.

Acknowledgments

This research was sponsored by the Air Force Flight Laboratory and the Air Force Office of Scientific Research.

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